

Binomial Model

Hull, Chapter 11

Additional reference:

John Cox and Mark Rubinstein, Options Markets, Chapter 5

1. One-Period Binomial Model

- Creating synthetic options (replicating options) by taking positions in the underlying asset and borrowing
- Pricing by replication
- Option's delta
- Deriving a one-period binomial option pricing formula and demonstrating that it does not depend on the real-world probabilities
- Risk-neutral probabilities
- Risk-neutral valuation from the no-arbitrage argument

2. Two-Period Binomial Model

- Extending the one-period model
- Deriving the two-period binomial option pricing formula

3. Multi-Step Cox-Ross-Rubinstein (CRR) Binomial Model

- Binomial model of asset price dynamics
- Working backwards through the binomial tree: the backward induction algorithm

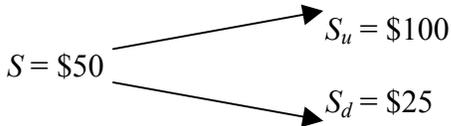
Binomial Model --- Introductory Example:

Current stock price: $S = \$50$

Call strike $K = \$50$

$r = 25\%$ per annum with annual compounding

In the next time period (one year) the stock can go either up to $S_u = \$100$ or down to $S_d = \$25$:



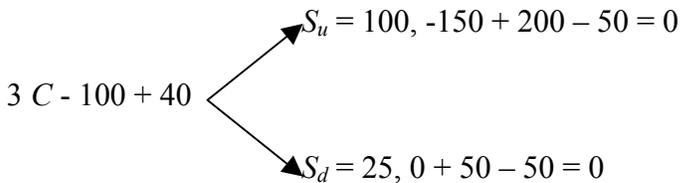
Question: Find the value of the call with $K = 50$.

The Power of No-Arbitrage Arguments:

If profitable arbitrage is not possible we can find the fair (arbitrage-free) value of the call C from the given data alone without any information about the probabilities of the stock going up or down!

Strategy:

1. Write 3 calls for C
2. Buy 2 shares for $S = \$50$
3. Borrow \$40 at 25% to be paid back at the end of the period (one year)



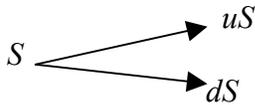
The future value of the portfolio is 0 in both cases independent of the outcome for the stock $S=100$ or 25, i.e., the portfolio is riskless and, assuming no arbitrage, its present value must also be zero:

$$3 C - 100 + 40 = 0 \quad \rightarrow \quad C = \$20.$$

If the call is not priced at \$20, arbitrage profits are possible.

To price the call we needed the following data:

K, S, r, T (time to maturity), range of the stock movement: u, d



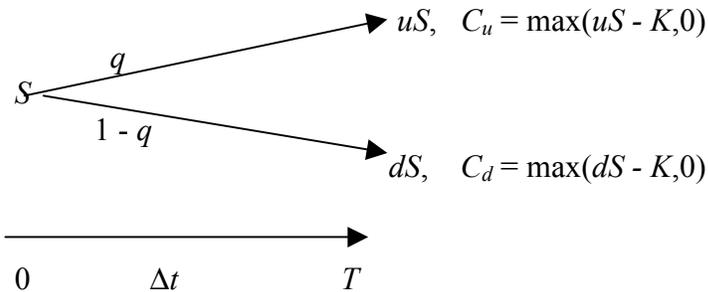
We did not need:

The probabilities q and $(1 - q)$ that the stock will go up or down.

We have priced the option *relative to the underlying stock and the risk-free rate*.

One Period Binomial Option Pricing: Replicating Portfolio

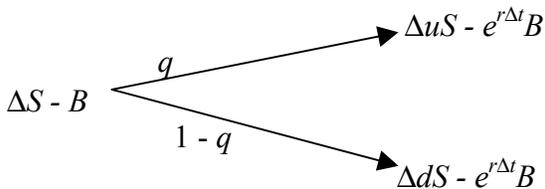
We assume the underlying asset pays no dividends (we will relax this assumption shortly). We also assume that $0 < d < e^{r\Delta t} < u$ (this is an important assumption)



Form the following portfolio (goal – replicate the option with positions in stock and borrowing):

- Borrow B dollars at risk-free rate (equivalently, short sell a zero-coupon risk-free bond with current price B and face value $e^{r\Delta t}B$ paid at the end of the period)
- Buy Δ shares of stock

Portfolio value at inception: $\Delta S - B$.



Let us select Δ and B so that the end-of-period values of our portfolio are equal to the call values for each possible outcome:

$$\Delta uS - e^{r\Delta t}B = C_u$$

$$\Delta dS - e^{r\Delta t}B = C_d$$

Solve these two equations for Δ and B :

$$\Delta = \frac{C_u - C_d}{(u - d)S} \quad (\text{Delta of the option})$$

$$B = e^{-r\Delta t} \left(\frac{dC_u - uC_d}{u - d} \right) \quad (\text{Amount to borrow})$$

Replicating portfolio (one period):

Thus we have *replicated* a call option with a long position in the underlying stock and borrowing (equivalently, short position in the bond). If the values of two portfolios are equal at the end of the period in all possible states \rightarrow they must be equal at the beginning of the period (*the no-arbitrage argument*):

$$C = \Delta S - B.$$

One-Period Binomial Pricing Formula

Substitute the expressions for Δ and B into the pricing formula:

$$\begin{aligned} C &= \Delta S - B \\ &= S \left(\frac{C_u - C_d}{(u - d)S} \right) - e^{-r\Delta t} \left(\frac{dC_u - uC_d}{u - d} \right) \\ &= e^{-r\Delta t} \left(\left(\frac{e^{r\Delta t} - d}{u - d} \right) C_u + \left(\frac{u - e^{r\Delta t}}{u - d} \right) C_d \right) \\ &= e^{-r\Delta t} [pC_u + (1 - p)C_d], \end{aligned}$$

where

$$p = \frac{e^{r\Delta t} - d}{u - d} \quad \text{and} \quad 1 - p = \frac{u - e^{r\Delta t}}{u - d}.$$

Just like pricing forwards, we constructed a *synthetic option (a replicating portfolio)* using the *no-arbitrage argument*.

The crucial feature of our result is that the probabilities q and $1 - q$ do not appear anywhere in the formula!

In this model, even if different investors have different beliefs about probabilities for the stock to go up or down, they still must agree on the relationship between the option price C , the stock price S and the risk-free rate of return r . At the same time, the call price relative to S and r does not depend on investors' attitudes toward risk. The discount rate is the risk-free rate r .

It should be stressed that the pricing formula is a *relative* pricing relationship that determines C , given S and r .

Risk-Neutral Probabilities:

The quantities p and $(1 - p)$ are called *risk-neutral probabilities*

$$p = \frac{e^{r\Delta t} - d}{u - d} \quad \text{and} \quad 1 - p = \frac{u - e^{r\Delta t}}{u - d}$$

As long as $0 < d < e^{r\Delta t} < u$, they are between 0 and 1.

Calculate expected stock price at the end of the period using risk-neutral probabilities:

$$puS + (1 - p)dS = \left[\left(\frac{e^{r\Delta t} - d}{u - d} \right) u + \left(\frac{u - e^{r\Delta t}}{u - d} \right) d \right] S = e^{r\Delta t} S$$

Thus, p and $(1 - p)$ are indeed *risk neutral probabilities*: expected rate of return on the stock under these probabilities is the risk-free rate r : $e^{r\Delta t} S$.

In the risk-neutral world, investors are risk-neutral and do not require any risk premium for holding risky assets. Expected rate of return on all assets is equal to the risk-free rate.

One-Period Risk-Neutral Valuation Formula

$$C = e^{-r\Delta t} [pC_u + (1 - p)C_d]$$

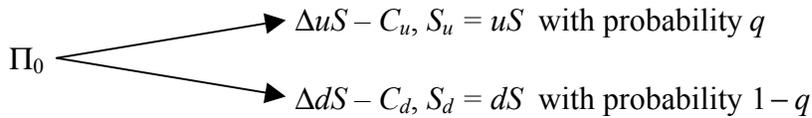
One-Period Binomial Option Pricing: Hedged Portfolio (alternative and equivalent derivation)

Set up a portfolio:

1. Short 1 call
2. Long Δ shares S

$$\Pi_0 = \Delta S - C$$

At time T :



For the portfolio to be riskless:

$$\Pi_u = \Pi_d, \quad \Delta u S - C_u = \Delta d S - C_d$$

There is no uncertainty about the future value of our portfolio: it is worth the same in *both states* (up state and down state).

$$\text{Solve for } \Delta: \quad \Delta = \frac{C_u - C_d}{(u - d)S}.$$

Delta is the ratio of the change in the option price to the change in the underlying price (*hedge ratio*).

Portfolio is riskless \rightarrow must earn interest at the risk-free rate, i.e.,

$$\Pi_0 = e^{-r\Delta t} \Pi_u,$$

$$\Delta S - C = e^{-r\Delta t} (\Delta u S - C_u),$$

$$C = \Delta S - e^{-r\Delta t} (\Delta u S - C_u),$$

$$\begin{aligned} C &= \left[\frac{C_u - C_d}{(u - d)S} \right] S(1 - ue^{-r\Delta t}) + e^{-r\Delta t} C_u \\ &= C_u \left(e^{-r\Delta t} + \frac{1 - ue^{-r\Delta t}}{u - d} \right) + C_d \left(\frac{ue^{-r\Delta t} - 1}{u - d} \right) \\ &= e^{-r\Delta t} [pC_u + (1 - p)C_d] \end{aligned}$$

In this derivation we hedged out all the risk of the short call position by buying Δ shares of the underlying (Δ is our hedge ratio). The resulting *hedged portfolio* is risk-free and as such must earn the risk-free rate of return to prevent arbitrage (no-arbitrage argument). This derivation of the option pricing based on the *hedged portfolio* is completely equivalent to the first derivation where we replicated a call with positions in the underlying and borrowing (*replicating portfolio*).

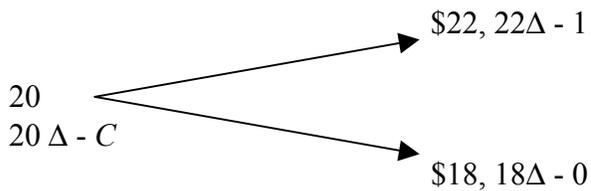
Example: One-Period Pricing

$$S = \$20$$

$$K = \$21$$

$$r = 12\%$$

$\Delta t = 0.25$ year



$$22\Delta - 1 = 18\Delta \rightarrow \Delta = 0.25$$

$$\Pi_0 = e^{-r\Delta t} \Pi_u,$$

$$\Pi_{,u} = 22 \times 0.25 - 1 = 4.5,$$

$$\Pi_0 = e^{-0.12 \times 0.25} 4.5 = 4.367$$

$$20 \times 0.25 - C = 4.367 \rightarrow C = 0.633$$

Existence and Uniqueness of Risk-Neutral Probabilities, Absence of Arbitrage Opportunities, and Market Completeness

The one-period binomial model contains one risky asset (a stock S) and one risk-free asset (one-period money market account or risk-free zero-coupon bond). More general models are multi-period (or continuous time) and include multiple risky assets, along with the risk-free asset (money market account). The very simple binomial model already contains many of the features of more general models.

- *Risk-neutral probabilities (risk-neutral probability measure)* exist if and only if there are no arbitrage opportunities in the model (this statement is known as *The First Fundamental Theorem of Asset Pricing*).
- A contingent claim is said to be *replicable* if it can be exactly replicated by an equivalent replicating portfolio of primitive underlying securities.
- A model is said to be *complete* if all contingent claims are replicable in this model.
- The risk-neutral probability measure is unique if and only if the market is complete (this statement is known as *The Second Fundamental Theorem of Asset Pricing*).

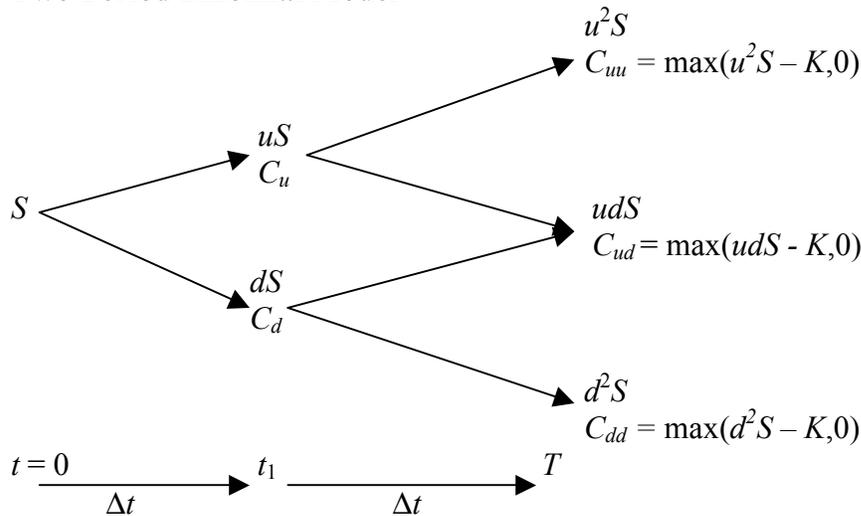
The binomial model with $0 < d < e^{r\Delta t} < u$ admits no arbitrage opportunities, is complete (any contingent claim can be replicated by positions in stock and money market account or bond), and there exists a unique risk-neutral probability measure (risk-neutral probabilities p and $1-p$).

Now suppose we started with a binomial model where $e^{r\Delta t} < d < u$. Then the stock return dominates the risk-free return in both states. Then there is an *arbitrage opportunity*: borrow cash and invest in stock. Initial value of this portfolio is zero, while the final value of this portfolio is positive in both states. The arbitrage portfolio is basically a

deterministic money machine. Clearly, this is not a realistic market model. This model admits arbitrage, and if you look at our risk-neutral probability formulas, you will see that $p < 0$ and $1 - p > 1$! Thus, in this case the risk-neutral probability measure does not exist.

Now suppose $0 < d < u < e^{r\Delta t}$. Then, the risk-free return dominates the return on the stock in both states. Then there is an *arbitrage opportunity*: sell the stock short and invest cash. Initial value of this portfolio is zero, while the final value of this portfolio is positive in both states. This model admits arbitrage, and has $p > 1$ and $1 - p < 0$. Thus, in this case there risk-neutral probability measure does not exist.

Two-Period Binomial Model



Two period option pricing formula

Two-period tree = combination of 3 one-period sub-trees:

$$C_u = e^{-r\Delta t}(pC_{uu} + (1 - p)C_{ud}),$$

$$C_d = e^{-r\Delta t}(pC_{ud} + (1 - p)C_{dd}),$$

$$C = e^{-r\Delta t}(pC_u + (1 - p)C_d)$$

$$= e^{-r(2\Delta t)}[p^2C_{uu} + 2p(1 - p)C_{ud} + (1 - p)^2C_{dd}]$$

Cox-Ross-Rubinstein (CRR) Multi-Period Binomial Model

$T = N\Delta t$; N time periods ($N + 1$ nodes along the path)

$N + 1$ different terminal prices

2^N possible price paths from $(0, S)$ to (T, S_T)

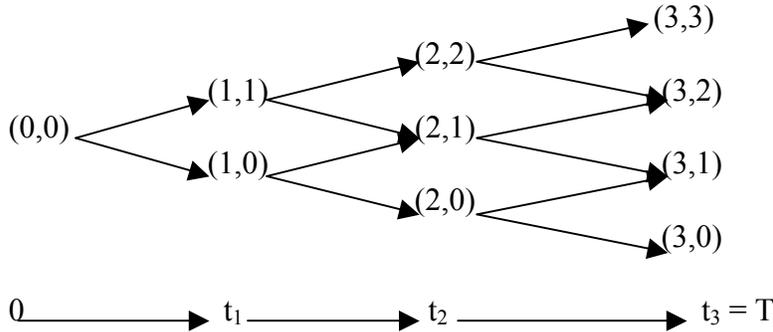
$S_{i,j} = u^j d^{i-j} S$: price at the node (i,j)

$i, i = 0, 1, \dots, N$: number of time steps to the node (i,j)

$j, j = 0, 1, \dots, i$: number of up moves to the node (i,j)

Payoff after N periods:

Call option payoff: $C_{N,j} = \max(u^j d^{N-j} S - K, 0)$



Working Backwards through the Tree: the Backward Induction Algorithm

Step N

$$C_{N,j} = \max(u^j d^{N-j} S - K, 0), j = 0, 1, \dots, N$$

Step $N-1$

$$C_{N-1,j} = e^{-r\Delta t} [p C_{N,j+1} + (1-p) C_{N,j}], j = 0, 1, \dots, N-1$$

...

Step i

$$C_{i,j} = e^{-r\Delta t} [p C_{i+1,j+1} + (1-p) C_{i+1,j}], j = 0, 1, \dots, i$$

...

Step 0

$$C = C_{0,0} = e^{-r\Delta t} [p C_{1,1} + (1-p) C_{1,0}]$$

Closed-form solution for European options

For European options, the algorithm can be solved in closed form:

$$C = e^{-rT} \sum_{j=0}^N \binom{N}{j} p^j (1-p)^{N-j} C_{N,j},$$

$C_{N,j} = \max(u^j d^{N-j} S - K, 0)$: terminal payoff at (N, j) ,

$e^{-rT} = e^{-r(N\Delta t)}$: N -period discount factor ($T = N\Delta t$),

$\binom{N}{j} = \frac{N!}{j!(N-j)!}$: number of paths to a particular outcome (N_j) at maturity (binomial coefficient),

$p_{N,j} = \binom{N}{j} p^j (1-p)^{N-j}$: risk-neutral probability of an outcome with j up movements and $N-j$ down movements (risk-neutral binomial probability).

Multi-Period Risk-Neutral Valuation Formula for European-style Contingent Claims with General Payoffs

Suppose a contingent claim is defined by its payoff $F(S_T)$, where F is a given function of the terminal asset price. Then its present value (price) is given by the discounted risk-neutral expectation of the payoff:

$$PV = e^{-rT} \sum_j p_{N,j} F(S_{N,j}),$$

where $p_{N,j}$ are risk-neutral probabilities.

Consistency check:

From the Binomial Theorem:

$$\begin{aligned} e^{-rT} \sum_{j=0}^N p_{N,j} S_{N,j} &= e^{-rT} \sum_{j=0}^N \binom{N}{j} p^j (1-p)^{N-j} u^j d^{N-j} S \\ &= e^{-rT} (pu + (1-p)d)^N S = e^{-rT} e^{rN\Delta t} S = S \end{aligned}$$

Thus, as expected, the price of the stock is equal to its current price.

American Options (No Dividends)

1. American calls: always worth more alive than dead. It is never optimal to exercise prior to expiration.
2. American puts: at each time step we maximize the value of our position -- optimal exercise strategy maximizes the option value to the option holder.

Assuming the rational option holder maximizes his portfolio value, the dynamic programming (Bellman) equation for American puts reads:

$$P_{i,j} = \max(K - u^j d^{i-j} S, e^{-r\Delta t} [pP_{i+1,j+1} + (1-p)P_{i+1,j}])$$

At each node we perform the test for making early exercise decision: is the option worth more dead (exercise) or alive (hold)?

Example of Delta-Hedging on a Binomial Tree: A Riskless Trading Strategy to Lock in Profits from Option Mispricing

From Cox and Rubinstein, Options Markets, Ch. 5

Three-period binomial tree ($N = 3$) for a call option with $K = 80$ expiring at the end of the third period. Initial stock price is $S = 80$ at the node $(0,0)$, the risk free rate r is such that $\exp(r\Delta t) = 1.1$, and the discount factors for one, two and three periods are:
 $\exp(-r\Delta t) = 0.909$, $\exp(-2r\Delta t) = 0.826$,
 $\exp(-3r\Delta t) = 0.751$.

Also, suppose that $u = 1.5$ and $d = 0.5$. The risk-neutral probability of an up move is:

$$p = \frac{1.1 - 0.5}{1.5 - 0.5} = 0.6$$

Fair value of the call according to our binomial model is $C = \$34.065$. Suppose that at time $t = 0$ the call is actually traded in the market for $C = \$36$, i.e., it is overpriced according to our model.

Arbitrage strategy: sell the call for \$36 and hedge by creating a synthetic call.

Step 0 ($i = 0$). Sell the call for 36. Take 34.065 of the proceeds and invest in a portfolio consisting of $\Delta = 0.719$ shares of stock by borrowing the difference $0.719 \times 80 - 34.065 = 23.455$.

Take the remainder $36 - 34.065 = 1.935$ and put it in the bank to grow at the risk free rate.

Step 1 ($i = 1$). Suppose the stock goes up to 120 so that the new $\Delta = 0.848$. Buy $0.848 - 0.719 = 0.129$ more shares at 120 per share for a total of 15.480. Borrow to pay for the stock. With the interest rate of 0.1, you already owe $23.455 \times 1.1 = 25.801$. Thus your total current debt is $25.801 + 15.480 = 41.281$.

Step 2 ($i = 2$). Suppose the stock now goes down to 60. The new $\Delta = 0.167$. Sell $0.848 - 0.167 = 0.681$ shares at 60 per share, taking in $0.681 \times 60 = 40.860$. Use it to pay back part of your debt. Since you now owe $1.1 \times 41.281 = 45.409$, the repayment will reduce this to $45.409 - 40.860 = 4.549$.

Step 3d ($i = 3$). Suppose the stock goes down to 30. The call you sold expired worthless. You own 0.167 shares of stock selling at 30 per share, for a total value of $0.167 \times 30 = 5$. Sell the stock and repay the debt $4.549 \times 1.1 = 5$ that you now owe.

Go back to the bank and withdraw your original deposit of 1.935, which has now grown to $1.935 \times (1.1)^3 = 2.575$. Congratulations! This is your arbitrage profit.

Step 3u ($i = 3$). Suppose that instead the stock price goes up to 90. The call you sold is in-the-money at expiration. Buy one share of stock and let the call be exercised, incurring a loss of $90 - 80 = 10$. You also own 0.167 shares of stock currently trading at 90/share, for a total value of $0.167 \times 90 = 15$. Sell the stock and use the proceeds to repay the debt and cover the loss of 10 for exercising the call.

Go back to the bank and withdraw your original deposit of 1.935, which has now grown to $1.935 \times (1.1)^3 = 2.575$. Congratulations! This is your arbitrage profit.

In summary, if our model of stock price movements is correct and we faithfully adjust our portfolio as prescribed by the formula for the option delta, then we can be assured of walking away in the clear at expiration date, while still keeping the original price difference plus the interest it has accumulated. In conducting the hedging operation, the essential thing was to maintain the proper proportional relationship: for each call we are short, we hold Δ shares of stock and we borrow at the risk free rate the dollar amount B , thus creating a *synthetic option*.

Notation:

Next to each node the following information is given:

Node indexes (i, j)

Risk neutral probability of getting to the node

S at the node

C at the node

Δ at the node (i, j) is computed according to
$$\Delta_{i,j} = \frac{C_{i+1,j+1} - C_{i+1,j}}{S_{i+1,j+1} - S_{i+1,j}} = \frac{C_{i+1,j+1} - C_{i+1,j}}{(u-d)S_{i,j}}$$

